MODULAR FORMS 2019: HECKE OPERATORS WEEK OF MAY 19, 2019

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1. Definition of Hecke operators

1.1. Overview.

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We recall Ramanujan's conjecture about the multiplicative relations in the coefficients of the modular discriminant $\Delta/(2\pi)^{12} = q+\sum_{n\geq 2}\tau(n)q^n$:

$$\tau(mn) = \tau(m)\tau(n), \quad \gcd(m,n) = 1$$

For instance, $\tau(2) = -24$, $\tau(3) = 252$, and $\tau(6) = -6048 = -24 \cdot 252$, so that $\tau(6) = \tau(2)\cdot\tau(3)$.

$$\tau(p)\tau(p^r) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1}), \quad p \text{ prime}, r \geq 1.$$

For instance $\tau(4) = -1472 = (-24)^2 - 2^{11} = \tau(2)^2 - 2^{11}$.

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These were proved by Mordell (1917), and can be combined as

$$\tau(m)\tau(n) = \sum_{d|\gcd(m,n)} d^{11}\tau(\frac{mn}{d^2})$$

What we will show that for on the space of cusp forms S_k ($k \geq 12$ even), there are linear operators (matrices) $T(n) : S_k \to S_k$ which

- $T(n)$ commute with each other,
- Satisfy the relations
$$T(m)T(n) = \sum_{d|\gcd(m,n)} d^{k-1}T(\frac{mn}{d^2})$$
- are self-adjoint with respect to the Petersson inner product on S_k, hence (by linear algebra) may be simultaneously diagonalized, that is there is an orthogonal basis of S_k consisting of joint eigenforms of all $T(n)$
- a joint eigenform f, with $T(n)f = \lambda(n)f$, $\forall n \geq 1$, has its Fourier coefficients given by
$$a_f(n) = a_f(1)\lambda_f(n)$$

Hence if f is a joint eigenform, then necessarily $a_f(1) = 0$, otherwise $f = 0$, and in that case we can normalize $a_f(1) = 1$, so that the Fourier expansion of f has coefficients

$$a_f(n) = \lambda_f(n), \quad a_f(1) = 1$$

and the Hecke eigenvalues $\lambda_f(n)$ inherits the multiplicative relations of $T(n)$, that is

$$\lambda_f(mn) = \sum_{d|\gcd(m,n)} d^{k-1}\lambda_f(\frac{mn}{d^2})$$

A special case is the modular discriminant $\Delta \in S_{12}$: Since S_{12} is one-dimensional, automatically Δ is a joint eigenform of all the $T(n)$, and hence the coefficients of the normalized form satisfy the Hecke relations, proving Ramanujan's conjectures.

1.2. **Definition of t_n and $T(n) = n^{k-1}t_n$.** We recall that a modular form of weight k is in particular given by a function F on the space of lattices, which is homogeneous degree $-k$: $F(\lambda L) = \lambda^{-k}F(L)$, $\lambda \in \mathbb{C}^*$. The recipe was to write for $\tau \in \mathbb{H}$, the lattice with positive basis $L = \langle \tau, 1 \rangle$

$$f(\tau) = F(\langle \tau, 1 \rangle)$$

We now define

$$t_nF(L) = \sum_{\substack{L' \subset L \\ [L:L']=n}} F(L')$$

as the sum over all sub-lattices of index n.

If F is homogeneous of degree $-k$, then so is T_nF, because the sub-lattices of index n in λL are precisely $\lambda L'$, where L' runs over all sub-pattices of index n in L, so that

$$t_nF(\lambda L) := \sum_{\substack{K\subset\lambda L\\ [L:K]=n}} F(K) = \sum_{\substack{L'\subset L\\ [L:L']=n}} F(\lambda L') = \sum_{\substack{L'\subset L\\ [L:L']=n}} \lambda^{-k}F(L') = \lambda^{-k}(t_nF)(L)$$

Hence t_n acts on the space of lattice functions of weight $-k$.

Next, we need to see the effect on the condition that $f(\tau)$ is holomorphic in $\mathbb{H}$, and that f is bounded at infinity. For this, it is convenient to first find an explicit parameterization of the sub-lattices of index n in a given lattice. It suffices to do so for the standard lattice:

Proposition 1.1. *Let L be a lattice with basis w_1, w_2: $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$. Then the sub-lattices of index n are*

$$L' = \langle aw_1 + bw_2, dw_2\rangle, \quad a, d \geq 1, \quad ad = n, \quad 0 \leq b < d$$

Proof. It suffices to show that the sub-lattices of index n in $\mathbb{Z}^2$ are $L' = g \cdot \mathbb{Z}^2$,

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{Z}), \quad ad = n, \quad a, d \geq 1, \quad 0 \leq b < d$$

that is

$$L' = \langle (a, b), (0, d)\rangle$$

This is essentially the Hermite normal form. Think of $\mathbb{Z}^2$ as row vectors, and take a basis of L' (of row vectors)

$$w_1' = (\alpha, \beta), \quad w_2' = (\gamma, \delta)$$

and note that (possibly after switching the two vectors)

$$\det(w_1' \mid w_2') = [\mathbb{Z}^2 : L'] = n.$$

Now apply row operations, of adding an integer multiple of one row to the order (so pre-multiplying by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$), performing the Euclidean algorithm for finding the GCD of the first column $\gcd(\alpha, \gamma) = a = x\alpha + y\gamma \geq 1$, until we end up with a new basis (it is still a basis since each step did not change this property)

$$(w_1'' \mid w_2'') = \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & d \\ a & b' \end{pmatrix}$$

Then if necessary we switch the vectors (and change one of their signs), which amounts to pre-multiplying the matrix of rows by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, to get a basis

$$\begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$$

Note that since we pre-multipled by an (integral) matrix, have not changed the determinant, so that $ad = n$. If necessary, now pre-multiply by $-I$ to achieve $a, d > 0$.

Finally, subtract a multiple of the second row from the first to replace b' by $b = b' - nd \in [0, d-1]$ to obtain a basis of the desired shape.

Example: We are given a basis of a sublattice of index 2 with row matrix

$$\begin{pmatrix} 6 & 4 \\ 5 & 3 \end{pmatrix}$$

which has determinant -2. Then switch rows to get a matrix with determinant $+2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix} = M_1$$

Then continue with row operations

$$M_1 \to \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} M_1 = \begin{pmatrix} 5 & 3 \\ 1 & 1 \end{pmatrix} = M_2 \to \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix} M_2 = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} = M_3$$
$$\to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M_3 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

so that

$$M = \begin{pmatrix} 6 & -1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

so that $L' = \langle (1,1), (0,2) \rangle$. □

Exercise 1. Let $L' = \{(x,y) \in \mathbb{Z}^2 : x + y = 0 \bmod 5\} \subset \mathbb{Z}^2$. Find the Hermite normal form $L' = \langle (a,b), (0,d) \rangle$ for L'

Consequently, the action of t_n on $f \in M_k$ which arises from the lattice function F, is by

$$(t_n f)(\tau) = \sum_{ad=n} \sum_{0 \le b < d} d^{-k} f(\frac{a\tau + b}{d})$$

Indeed,

$$\begin{aligned}(t_n f)(\tau) &= \sum_{ad=n} \sum_{0 \le b < d} F\Big(\langle a\tau + b, d \rangle\Big) \\ &= \sum_{ad=n} \sum_{0 \le b < d} d^{-k} F\Big(\langle \frac{a\tau + b}{d}, 1 \rangle\Big) = \sum_{ad=n} \sum_{0 \le b < d} d^{-k} f(\frac{a\tau + b}{d})\end{aligned}$$

We introduce a different normalization of the Hecke operators

$$T(n) = n^{k-1} t_n$$

which will result in cleaner formulas. Thus for $f \in M_k$,

$$T(n)f(\tau) = n^{k-1} \sum_{ad=n} \sum_{0 \le b < d} d^{-k} f(\frac{a\tau + b}{d}) \tag{1}$$

From (1) we see that if $f(\tau)$ is analytic in τ then so is $T(n)f$; and if f is bounded at infinity then so is $T(n)f$ in fact

$$T(n)f(i\infty) = n^{k-1} \sum_{ad=n} \sum_{0 \le b < d} d^{-k} f(i\infty) = n^{k-1} \sum_{d|n} d^{1-k} f(i\infty) = \sigma_{k-1}(n) f(i\infty)$$

and in particular $T(n)$ preserves the space of cusp forms: $T(n) : S_k \to S_k$.

We summarize all this by

Theorem 1.2. *The Hecke operators act on M_k and preserve the space of cusp forms S_k.*

1.3. Action on Fourier coefficients.

Proposition 1.3. *Assume $f \in S_k$ has Fourier expansion*

$$f(\tau) = \sum_{m \ge 1} A(m) q^m$$

Then $T(n)f$ has the expansion $T(n)f = \sum_{m \ge 1} B_n(m) q^m$ with

$$B_n(m) = \sum_{a | \gcd(m,n)} a^{k-1} A(\frac{mn}{a^2})$$

Proof. Setting $e(z) = e^{2\pi i z}$, we have

$$\begin{aligned} T(n)f &= n^{k-1} \sum_{m \ge 1} A(m) \sum_{ad=n} \sum_{0 \le b < d} d^{-k} e(m \frac{a\tau + b}{d}) \\ &= n^{k-1} \sum_{m \ge 1} A(m) \sum_{ad=n} d^{-k} e(\frac{ma\tau}{d}) \sum_{0 \le b < d} e(\frac{mb}{d}) \end{aligned}$$

Now

$$\sum_{0 \le b < d} e(\frac{mb}{d}) = \begin{cases} d, & d \mid m \\ 0, & d \nmid m \end{cases}$$

so that

$$T(n)f = n^{k-1} \sum_{m \ge 1} A(m) \sum_{\substack{ad=n \\ d|m}} d^{1-k} e(m \frac{a\tau}{d})$$

Writing $m = dm'$ this becomes

$$T(n)f = \sum_{m' \geq 1} A(dm') \sum_{ad=n} (\frac{n}{d})^{k-1} q^{m'a}$$

Collecting together powers of q, by setting $m'' = am'$ and writing $d = n/a$, we obtain

$$T(n)f = \sum_{m'' \geq 1} q^{m''} \sum_{a|m'', a|n} a^{k-1} A(\frac{m''n}{a^2})$$

and therefore

$$B_n(m) = \sum_{a|\gcd(m,n)} a^{k-1} A(\frac{mn}{a^2})$$

□

1.4. **Eigenforms.** Now assume that f is a joint eigenfunction of all the $T(n)$'s:

$$T(n)f = \lambda(n)f, \quad \forall n \geq 1$$

Then $B_n(m) = \lambda(n)A(m)$, so that

$$\lambda(n)A(m) = \sum_{a|\gcd(m,n)} a^{k-1} A(\frac{mn}{a^2})$$

Taking $m = 1$ gives

$$\lambda(n)A(1) = A(n)$$

In particular, for an eigenform $f \neq 0$ we must have $\boxed{A_f(1) \neq 0}$. Hence we can normalize by $A_f(1) = 1$, and then we obtain

$$f = q + \sum_{n \geq 2} \lambda_f(n) q^n, \quad T(n)f = \lambda_f(n)f$$

Corollary 1.4. *The Ramanujan tau function satisfies the Hecke relations conjectured by Ramanujan.*

Proof. Since $S_{12} = \mathbb{C}\Delta$ is one-dimensional, it is automatic that Δ is a simultaneous eigenform of all Hecke operators, and hence

$$\tau(n) = \lambda_\Delta(n)$$

Therefore $\tau(n)$ satisfies the relations

$$\tau(m)\tau(n) = \sum_{d|\gcd(m,n)} d^{12-1} \tau(\frac{mn}{d^2})$$

□

1.5. Commutativity, Hecke relations and self-adjointness.

Proposition 1.5. *The operators $T(n)$ on M_k satisfy the relations*

$$T(m)T(n) = T(mn), \quad \gcd(m,n) = 1$$

$$T(p)T(p^r) = T(p^{r+1}) + p^{k-1}T(p^{r-1})$$

Corollary 1.6. *All Hecke operators commute with each other.*

Recall the Petersson inner product on S_k:

$$\langle f, g\rangle = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}y^k \frac{dxdy}{y^2}$$

Theorem 1.7. *The operators $T(n)$ are self-adjoint with respect to the Petersson inner product:*

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle, \quad f, g \in S_k$$

Since $T(n)$ are a commuting family of self-adjoint operators on S_k, by a basic fact of linear algebra they may be simultaneously diagonalized. Hence

Corollary 1.8. *There is an orthogonal basis of S_k consisting of joint eigenforms of all $T(n)$.*

This basis is unique, as we see below.

1.6. Multiplicity one.

As a corollary, we deduce a "multiplicity one" statement:

Corollary 1.9. *If two nonzero eigenforms $f, g \in S_k$ have the same Hecke eigenvalue for all n, then they must by scalar multiples of each other: $f = cg$, $c \in \mathbb{C}^*$.*

Indeed, if $\lambda_f(n) = \lambda_g(n)$ for all $n \geq 1$ then we have $f = a_f(1)\sum_{n\geq 1}\lambda(n)q^n$, $g = a_g(1)\sum_{n\geq 1}\lambda(n)q^n$ so that $g = \frac{a_g(1)}{a_f(1)}f$.

Exercise 2. Suppose $f = A(0) + \sum_{m\geq 1} A(m)q^m \in M_k$ is a non-cuspidal modular form of weight k (so $A(0) \neq 0$). Show that $T(n)f = \sum_{m\geq 0} B_n(m)q^m$ with

$$B_n(0) = A(0)\sigma_{k-1}(n)$$

and

$$B_n(m) = \sum_{d|\gcd(m,n)} d^{k-1}A(\frac{mn}{d^2}), \quad m \geq 1$$

Deduce that if such f is an eigenform of all Hecke operators $T(n)f = \lambda_f(n)f$, $n \geq 1$, then $\lambda_f(n) = \sigma_{k-1}(n)$ and $f = cE_k$, with $c = A(1)/\gamma_k$,

$E_k = 1 + \gamma_k \sum_{m\geq 1} \sigma_{k-1}(m) q^m$ being the normalized Eisenstein series. Thus there is essentially only one non-cuspidal Hecke eigenform in M_k.

Exercise 3. If $f \in S_k$ is a Hecke eigenform, with Hecke eigenvalue $T(n)f = \lambda_f(n)f$, show that for prime p, and $|X| < p^{-1}$,

$$\sum_{j=0}^{\infty} \lambda_f(p^j) X^j = \frac{1}{1 - \lambda_f(p)X + p^{k-1}X^2}$$

1.6.1. *Maeda's conjecture.* A strong form of the Maeda's conjecture states that for $n > 1$, the characteristic polynomial of the linear map $T(n)$ on S_k is irreducible. This has been checked up to very large weights.